Deterministic Polynomial Identity Tests for Multilinear Bounded-Read Formulae

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Arithmetic Formula Identity Testing

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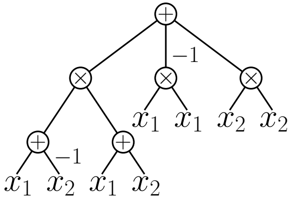

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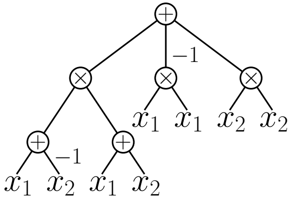

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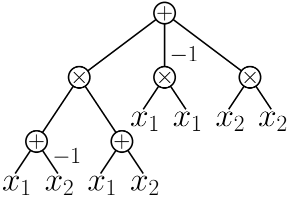

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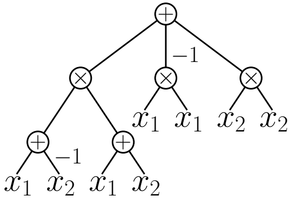

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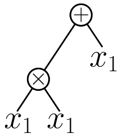

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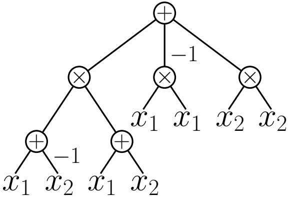

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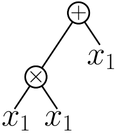

$$x_1^2 + x_1 \not\equiv 0$$

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Randomized algorithm:

Pick $a_i \in_U S \subseteq \mathbb{F}$ uniformly, ACCEPT iff $P(a_1, \ldots, a_n) = 0$.

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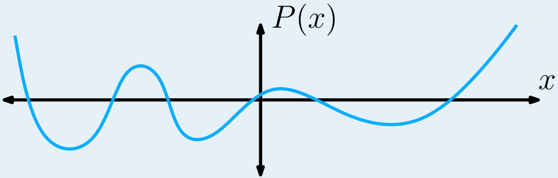

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- It is a next natural candidate problem to derandomize.

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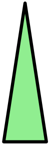

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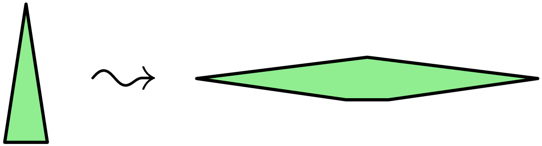

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Main Theorem

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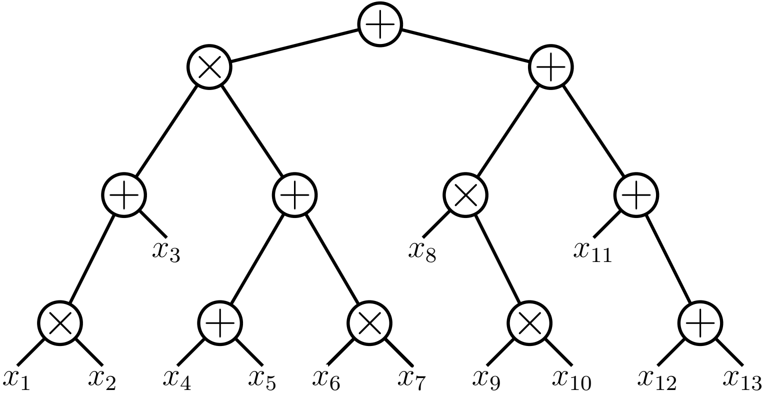

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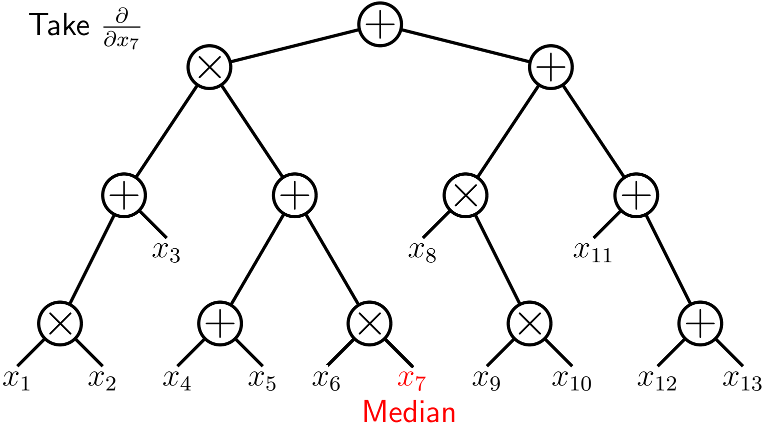

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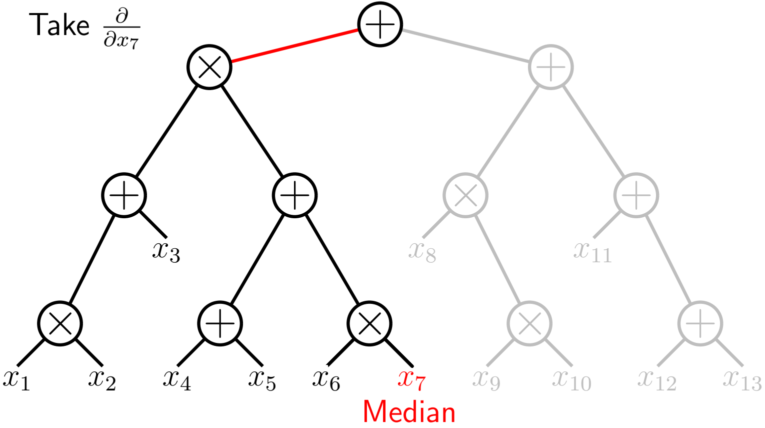

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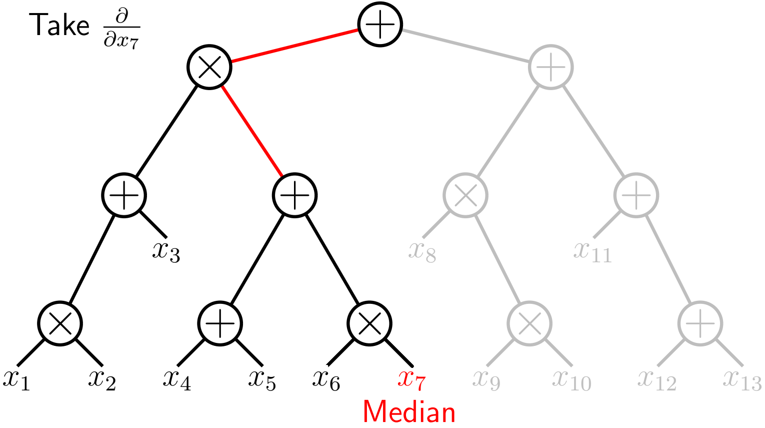

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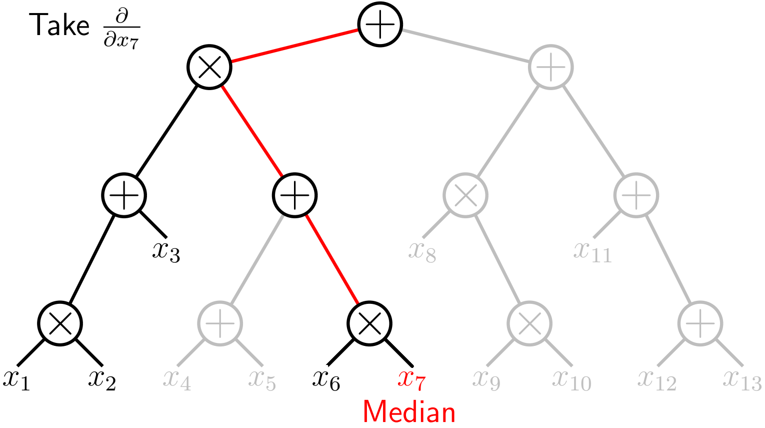

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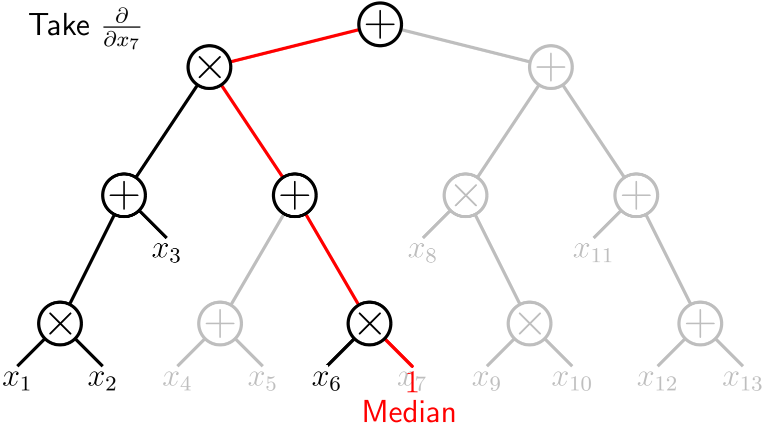

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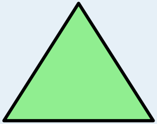

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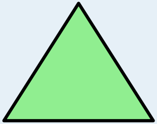

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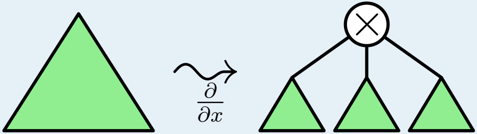

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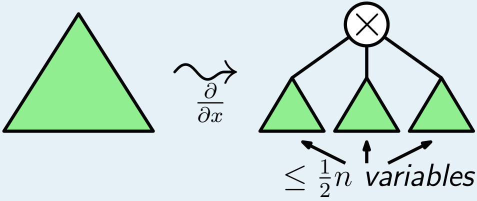

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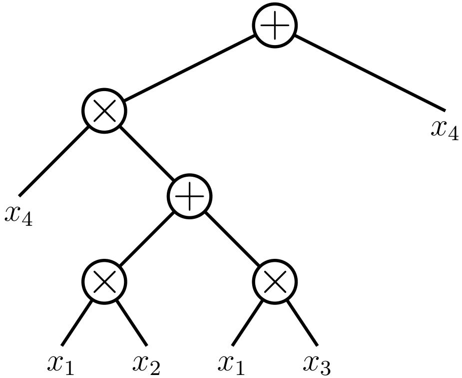

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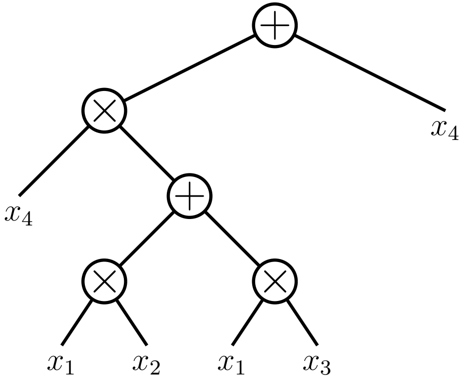

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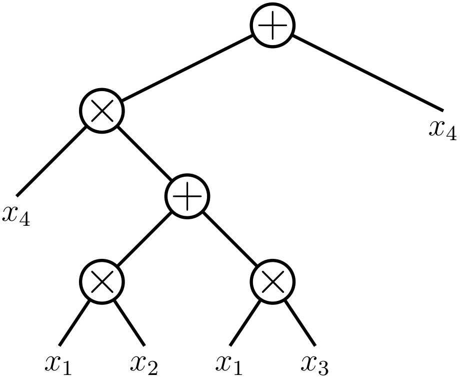

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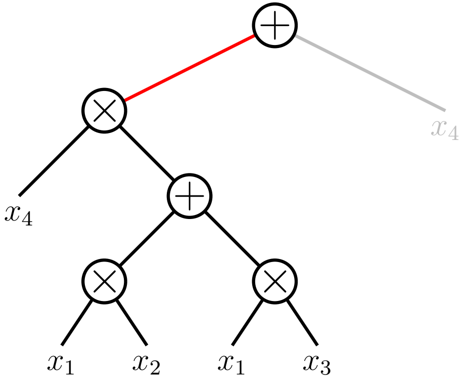

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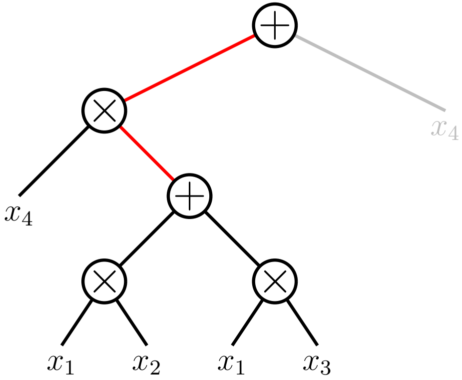

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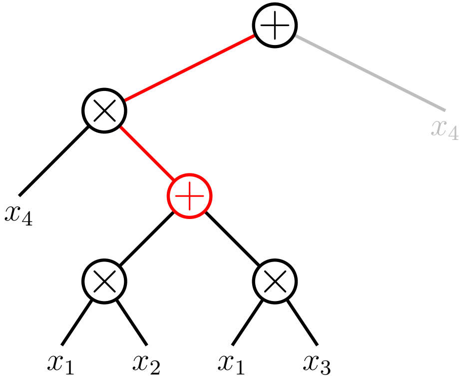

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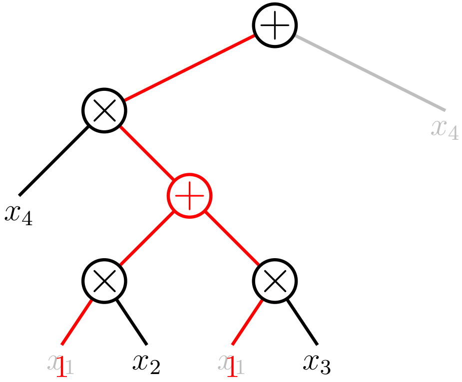

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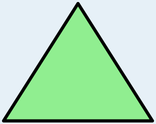

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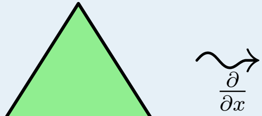

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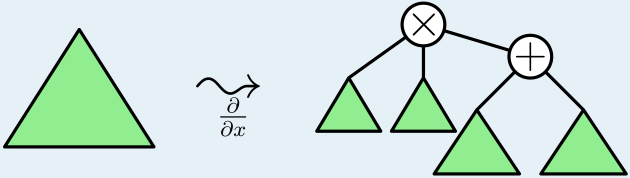

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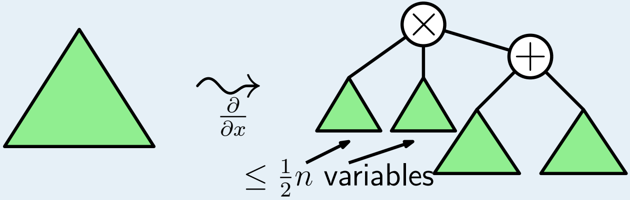

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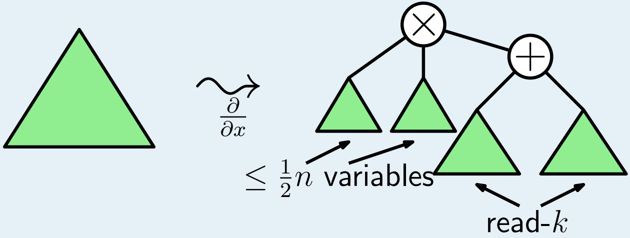

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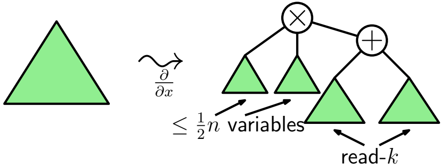

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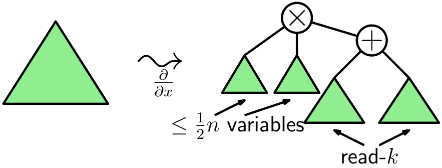

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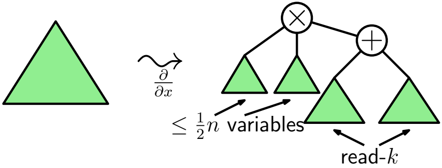

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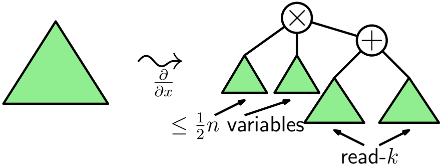

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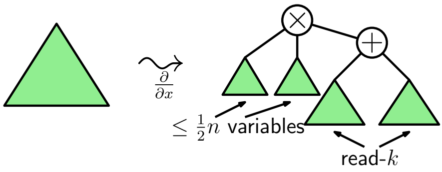
ACCEPT iff $F = 0$.



The First Reduction

Read-$(k+1) \leq \sum^2$-Read-k

While F has variables:

- Select x which fragments F.
- Test the factors of $\partial_x F$ recursively.
- If all factors of $\partial_x F$ are nonzero, REJECT.
- Set $F = F|_{x \leftarrow 0}$.

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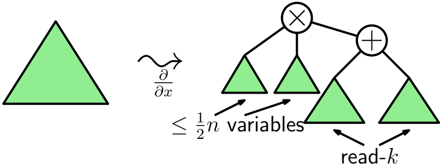

Makes $n^{O(\log n)}$ calls to the $\sum^2$-read-k identity test.

Outline

Weakened Main Theorem

There is a $s^{O(1)} \cdot n^{k^{O(k)}+O(k \log n)}$ time deterministic identity test for size-s n-variable multilinear read-k formulae.

Techniques:

1. **Fragmenting**

   Reduces multilinear read-$(k+1)$ to multilinear $\sum^2$-read-k.

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Testing $\sum^2$-read-k $\leq$ Testing read-k

Fact (SV Hitting Set, implicit in [SV09])

The set of binary strings H_w with Hamming weight at most w hits any class $\mathcal{F}$ of multilinear polynomials that:

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- For such a $\bar{\sigma}$, $H_w + \bar{\sigma}$ hits F.

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Witness Lemma

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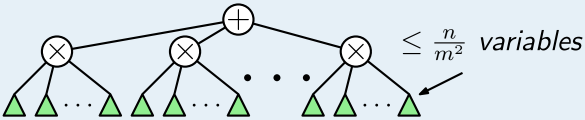

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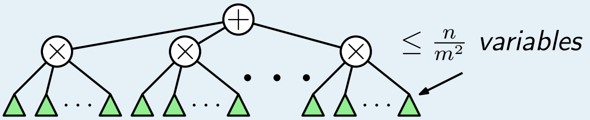

then F does not compute a monomial of degree n.

The Shattering Lemma

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For any nonzero multilinear $\sum^2$*-read-*k *formula* F *on* n *variables, there exist sets of variables*

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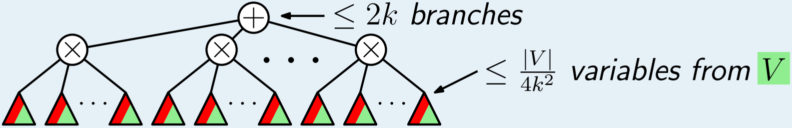

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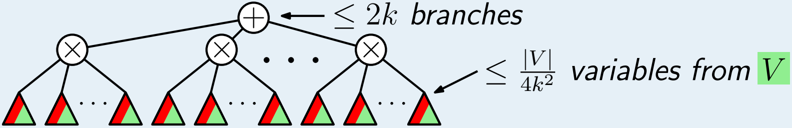

where each small subformula is the partial derivative of some subformula of F.

Theorem (Correctness)

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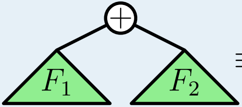

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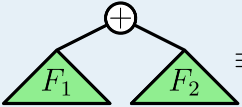
$\Rightarrow$ **Shatter(** 

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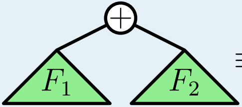
$\Rightarrow$ **Shatter(** 

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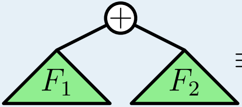
$\Rightarrow$ **Shatter(** 

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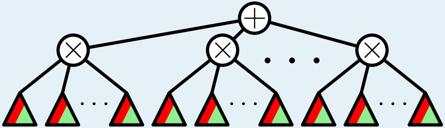
$\overset{\partial_P}{\Rightarrow}$  $\equiv M_{n-|P|}$

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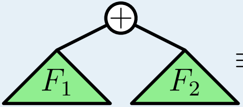

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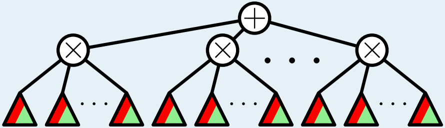

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Set vars outside V.

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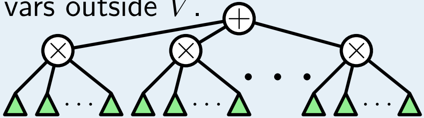

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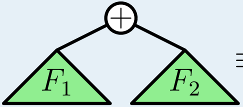
$\Rightarrow$ **Shatter**(

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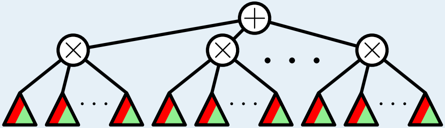
$\overset{\partial_P}{\Rightarrow}$  $\equiv M_{n-|P|}$

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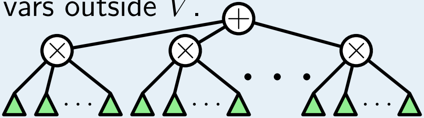
$\Rightarrow$  $\equiv M_{|V|}, |V| \geq \frac{n}{k^{O(k)}} \geq 1$

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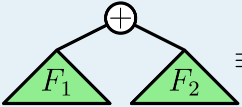

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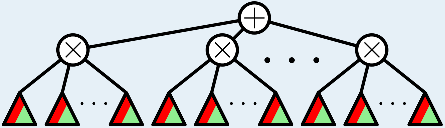

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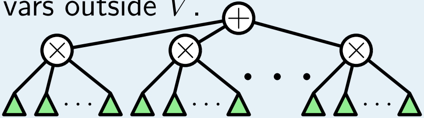

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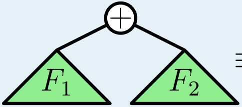
$\Rightarrow$ **Shatter(** 

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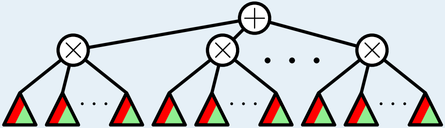
$\Rightarrow$ 
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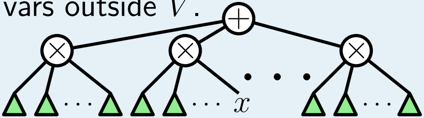
$\Rightarrow$ 

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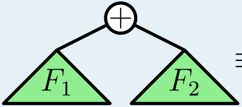

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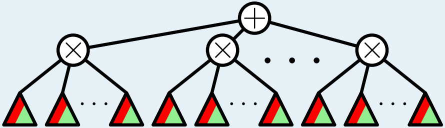

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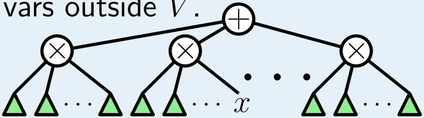

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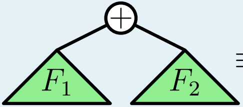

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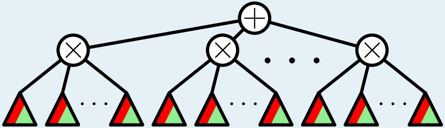

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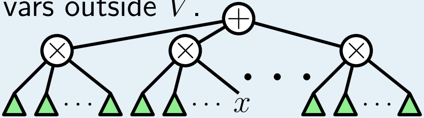

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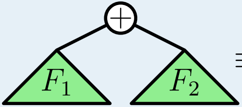

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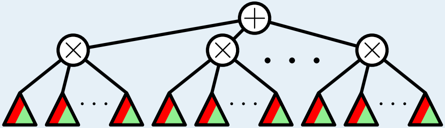

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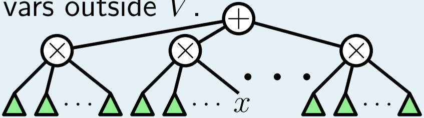

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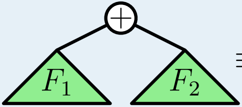

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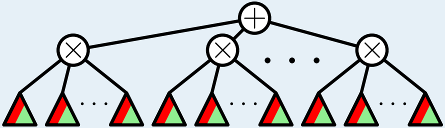

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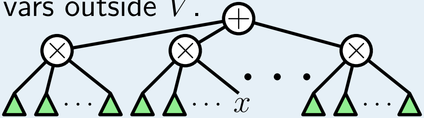

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Determine $\bar{\sigma}$ using trial substitution and read-k identity test. ∎

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   Reduces multilinear $\sum^2$-read-k to multilinear read-k.

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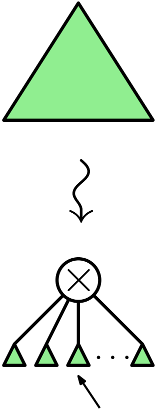
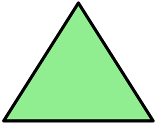


$\leq \alpha n$ variables



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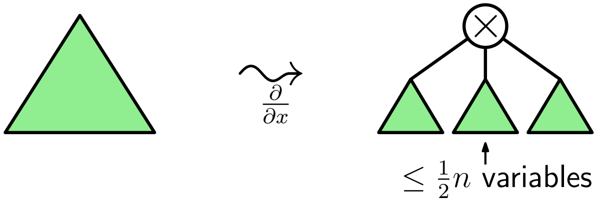

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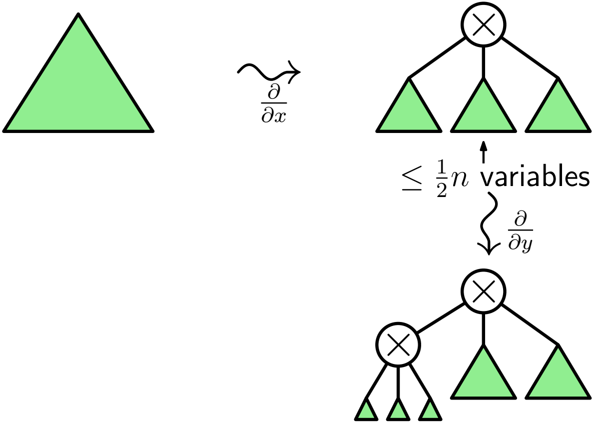

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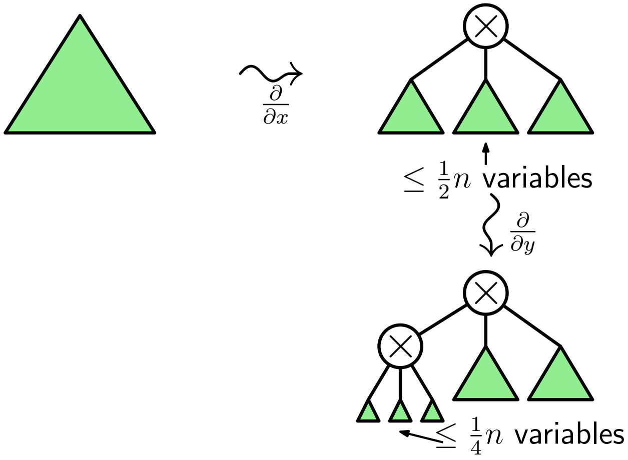

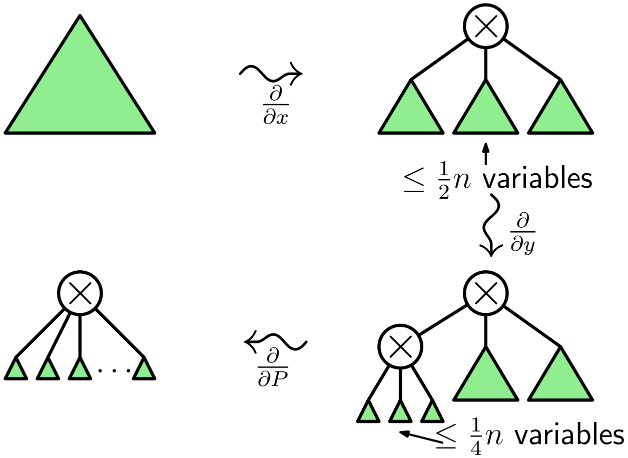
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$\leq \frac{1}{2}n$ variables
$\frac{\partial}{\partial y}$
$\frac{\partial}{\partial P}$
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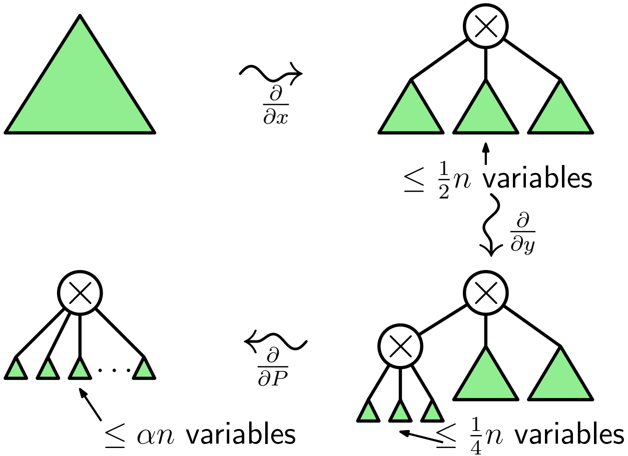

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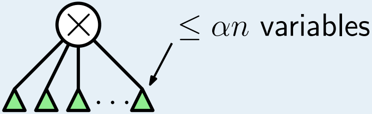
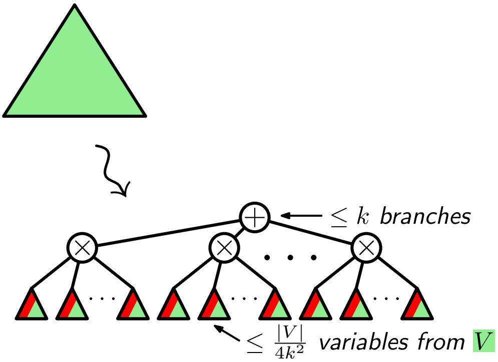
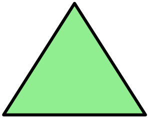


Shattering Read-k Formulae





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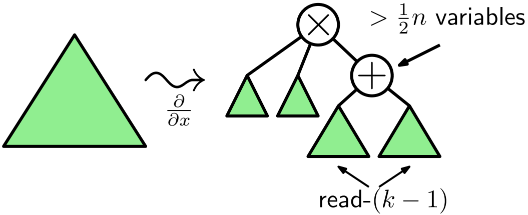

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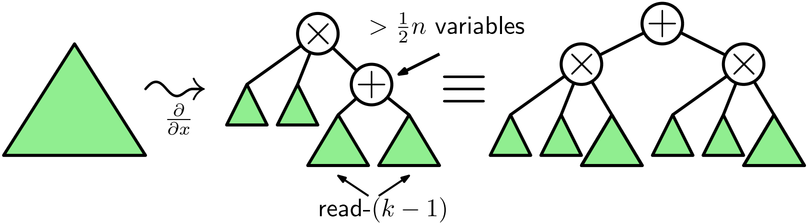

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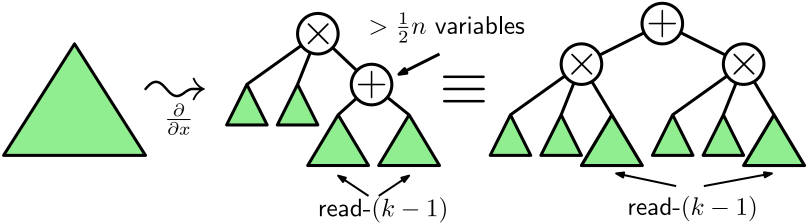

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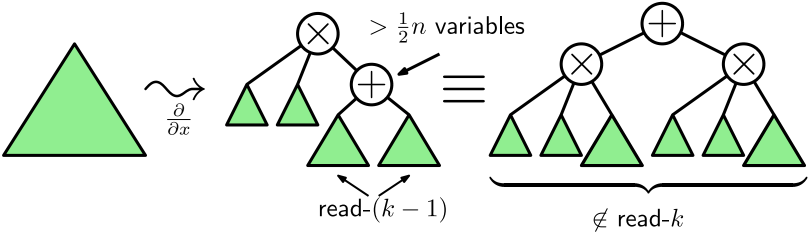

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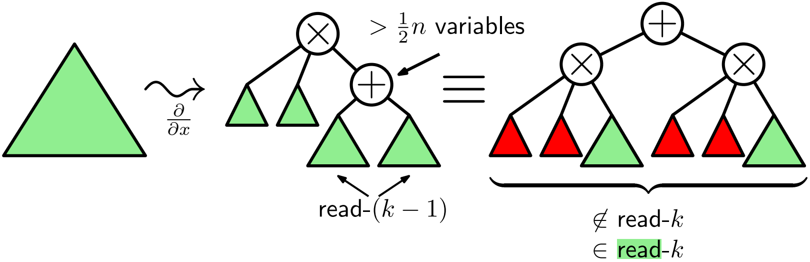

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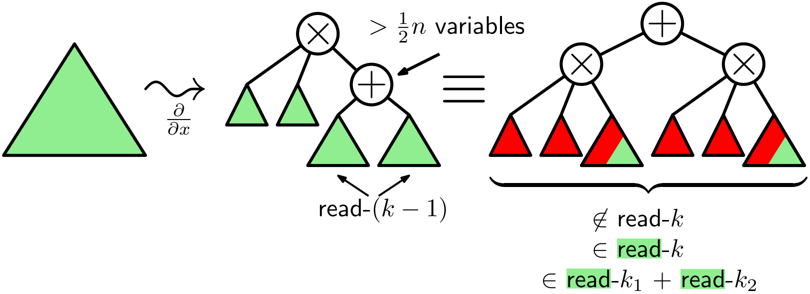

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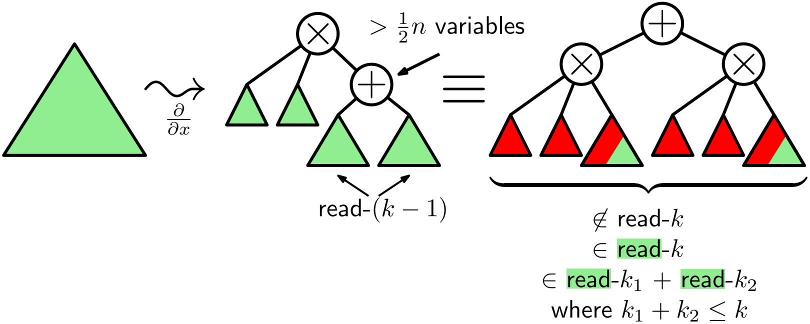

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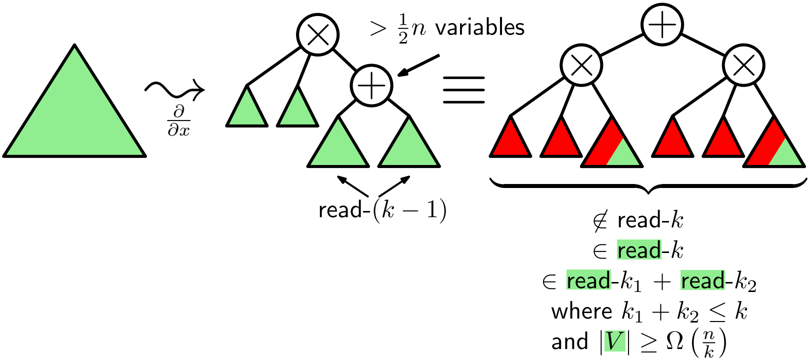

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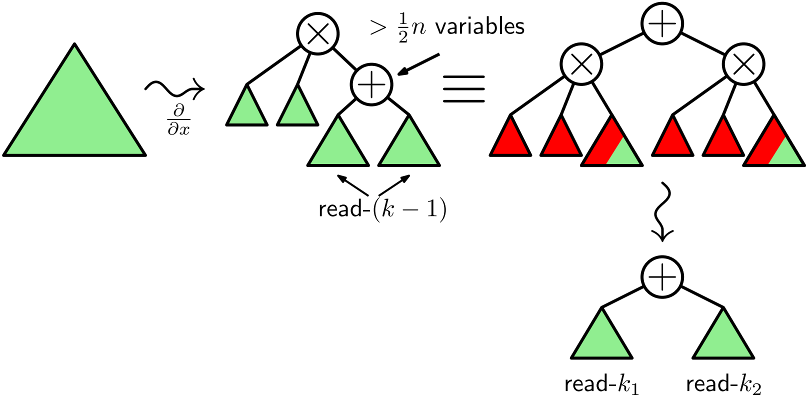

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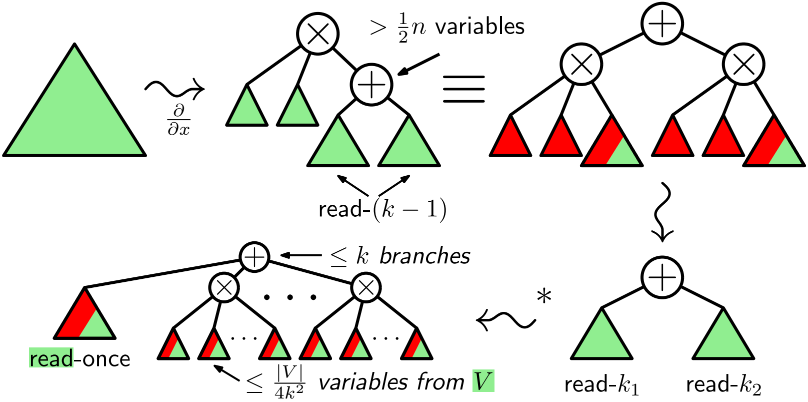

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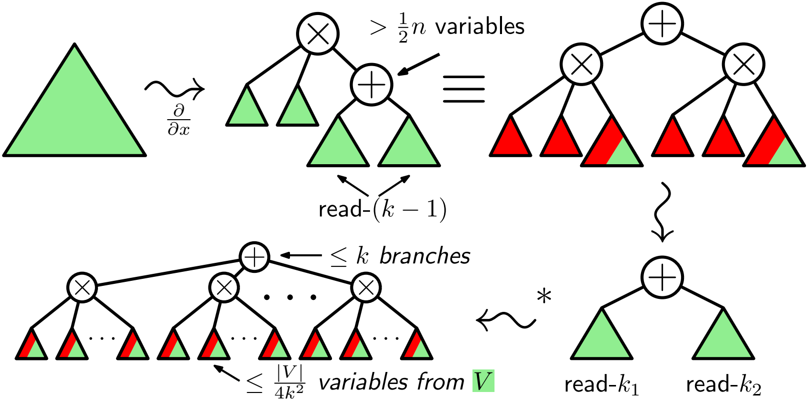

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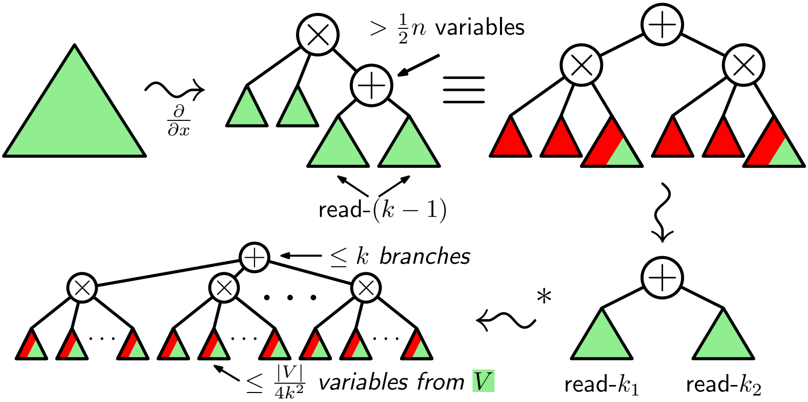


At most k iterations are required to successfully shatter a read-k formula.

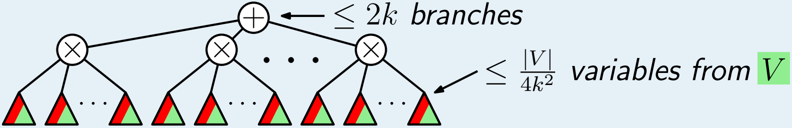
The Shattering Lemma

Shattering Lemma

For any nonzero multilinear $\sum^2$*-read-*k *formula* F *on* n *variables, there exist sets of variables*

- P, *with* $|P| = \mathrm{poly}(k)$, *and*
- V, *with* $|V| = \frac{n}{k^{O(k)}}$

such that $\frac{\partial F}{\partial P}$ *depends on at least the variables in* V, *and can be written as*



where each small subformula is the partial derivative of some subformula of F.

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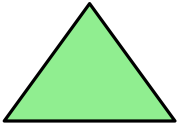

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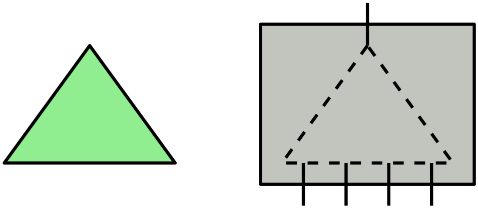

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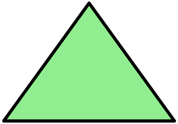

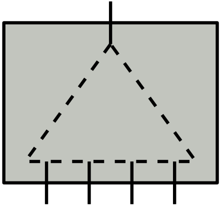

Theorem (Agrawal-Vinay)

*A **blackbox** poly-time identity test for depth-4 formula implies a blackbox subexp-time identity test for arithmetic formula.*

Extension: Blackbox - Outline

- Hitting Set Generators
- SV Generator
- Making our algorithm blackbox

Extension: Blackbox - Hitting Set Generators

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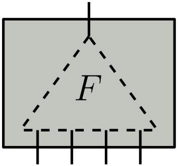

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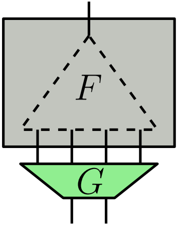

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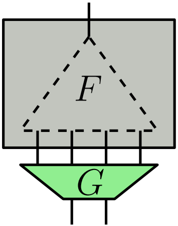

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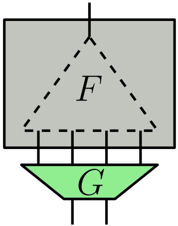

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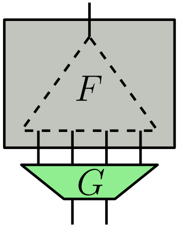
Efficient HSGs: low degree d_G and seed length m.



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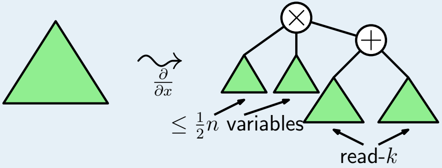

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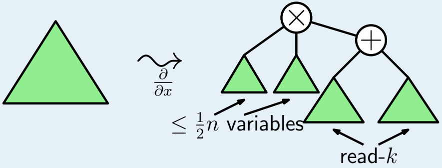

- By IH, $G + G_{SV}^{(\log n)-1}$ is a HSG for $\partial_x F$.

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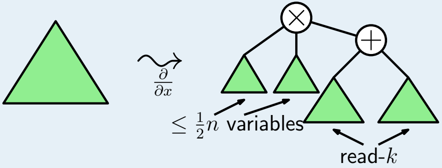

- By IH, $G + G_{SV}^{(\log n)-1}$ is a HSG for $\partial_x F$.
- Therefore, $G + G_{SV}^{\log n}$ is a HSG for F. ∎

Extension: Blackbox - $\sum^2$-Read-$k \leq$ Read-k

Lemma

If G is a HSG for read-k formulae, then $G + G_{SV}^{k^{O(k)}}$ is a HSG for $\sum^2$-read-k formulae.

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- Apply the non-blackbox analysis. ∎

Extension: Blackbox - The Final HSG

Theorem

$G_{SV}^{k^{O(k)}+O(k\log n)}$ *is a HSG for multilinear read-k formula.*

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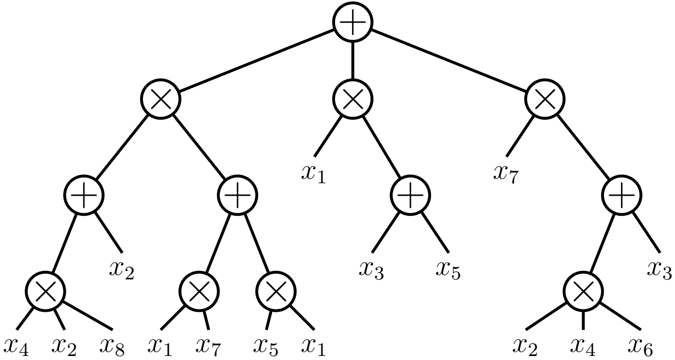
There is a polynomial-time blackbox identity test for multilinear constant-read constant-depth formulae.

Idea: *Analyze the depth parameter in the Fragmentation Lemma.*

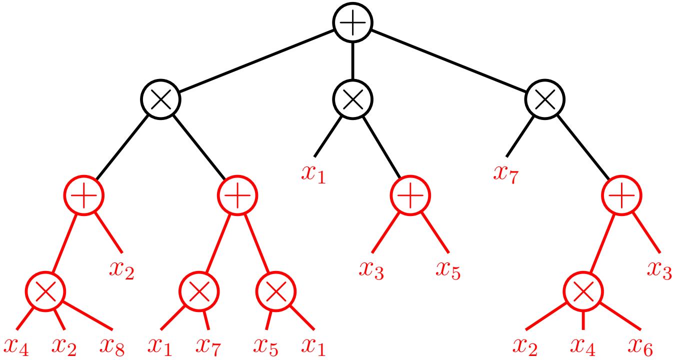
Extension: Sparse-Substituted

Read-3 depth-4



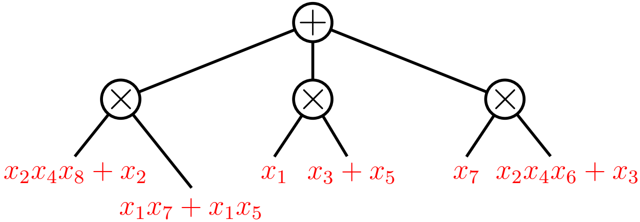
Extension: Sparse-Substituted

Read-3 depth-4



Extension: Sparse-Substituted

Read-3 depth-4 (and read-2 depth-2 sparse-substituted)



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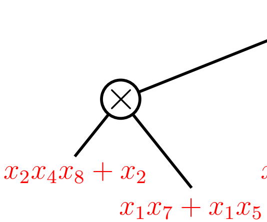

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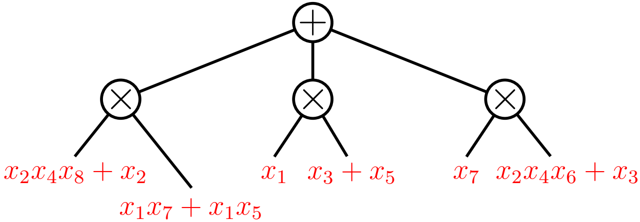

- Our tests extend to this model at quasi-polynomial cost.

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- Encompasses tests for
  - Multilinear Constant-Top-Fanin Depth-4 [KMSV10],
  - A generalized version of $\sum^k$-Read-Once [SV09].

Summary

Main Theorem

There is a polynomial-time deterministic identity test for multilinear constant-read formulae.

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Extensions:

1. Blackbox: quasi-poly-time.
2. Sparse substituted: quasi-poly-time.

Open Questions

- Is there a poly-time blackbox test for multilinear constant-read formulae?
- Can we drop the multilinearity requirement?
- For these types of formulae can we get
  - interesting lower bounds?
  - reconstruction algorithms?
- Is AFIT in P?
- Can any randomized algorithm be efficiently derandomized?

Questions?

Thanks!